SOME BINOMIAL-COEFFICIENT SUMS

For nonnegative integers n, k, let

$$P(n,k)=\int_0^1 (1-x)^n x^k dx. \tag{1}$$

Evidently $P(0,k)=P(k,0)=\frac{1}{k+1}$. If n is positive, integration by parts gives

$$\begin{aligned} P(n,k) &= \int_0^1 (1-x)^n x^k dx \\ &= \frac{1}{k+1}(1-x)^n x^{k+1}\Big|_0^1 + \int_0^1 \frac{n}{k+1}(1-x)^{n-1}x^{k+1}dx \\ &= 0+\frac{n}{k+1}P(n-1,k+1) \end{aligned}$$

Hence

$$\begin{aligned} P(n,k) &= \frac{n}{k+1}P(n-1,k+1) \\ &= \frac{n(n-1)}{(k+1)(k+2)}P(n-2,k+2) \\ &= \cdots = \frac{n!}{(k+1)(k+2)\cdots(k+n)}P(0,k+n) \\ &= \frac{n!}{(k+1)(k+2)\cdots(k+n)(k+n+1)} \\ &= \frac{1}{k+n+1}\frac{n!k!}{(n+k)!} \\ &= \frac{1}{k+n+1}\binom{n+k}{k}^{-1}. \end{aligned}$$

On the other hand, expanding out $(1-x)^n$ in (1) gives

$$P(n,k)=\int_0^1 \sum_{j=0}^n \binom{n}{j}(-1)^j x^{j+k}dx = \sum_{j=0}^n \frac{(-1)^j}{k+j+1}\binom{n}{j}.$$

Thus, we have the identity

$$\sum_{j=0}^n \frac{(-1)^j}{k+j+1}\binom{n}{j} = \frac{1}{k+n+1}\binom{n+k}{k}^{-1} \tag{2}$$

for all nonnegative integers n and k. The case $k = 0$ is

$$\sum_{j=1}^{n} \frac{(-1)^j}{j+1}\binom{n}{j} = \frac{1}{n+1},$$

as proved earlier. Of course (2) does not hold for $k = -1$, but in that case we have

$$\sum_{j=0}^{n} \frac{(-1)^j}{j}\binom{n}{j} = -\int_0^1 \frac{u^n - 1}{u-1}du = -H_n.$$

Exercise 1. Define for nonnegative integers k, n

$$Q(n,k) = \int_0^1 (1+x)^n x^k dx.$$

a. Show that

$$Q(n,k) = \sum_{j=0}^{n} \frac{1}{k+j+1}\binom{n}{j}.$$

b. Using integration by parts, show that for $k > 0$

$$Q(n,k) = \frac{2^{n+1}}{n+1} - \frac{k}{n+1}Q(n+1, k-1).$$

c. Explain why $Q(n,0) = (2^{n+1} - 1)/(n+1)$, and use this fact together with part (b) to get a formula for $Q(n,1)$.

Suppose now we want to evaluate

$$\sum_{j=0}^{n} \frac{(-1)^j}{2j+1}\binom{n}{j}.$$

We have

$$\sum_{j=0}^{n} (-1)^j x^{2j}\binom{n}{j} = (1-x^2)^n$$

and so

$$\sum_{j=0}^{n} \frac{(-1)^j}{2j+1}\binom{n}{j} = \int_0^1 (1-x^2)^n dx.$$

On the other hand, $I(n) = \int_0^1 (1-x^2)^n dx$ can be written

$$I(n) = \int_0^{\frac{\pi}{2}} (1-\sin^2\theta)^n \cos\theta d\theta = \int_0^{\frac{\pi}{2}} \cos^{2n+1}\theta d\theta$$

via the substitution $x = \sin\theta$, and so, integrating by parts,

$$I(n) = \int_0^{\frac{\pi}{2}} \cos^{2n}\theta \cos\theta d\theta = \cos^{2n}\theta \sin\theta\Big|_0^{\frac{\pi}{2}} + 2n\int_0^{\frac{\pi}{2}} \cos^{2n-1}\theta \sin^2\theta d\theta$$
$$= 0 + 2n\int_0^{\frac{\pi}{2}} (\cos^{2n-1}\theta - \cos^{2n+1}\theta) d\theta = 2n(I(n-1) - I(n)).$$

Add $2nI(n)$ to both sides to get $(2n+1)I(n) = 2nI(n-1)$, or

$$I(n) = \frac{2n}{2n+1} I(n-1).$$

Hence

$$\begin{aligned} I(n) &= \frac{2n}{2n+1} I(n-1) \\ &= \frac{(2n)(2n-2)}{(2n+1)(2n-1)} I(n-2) \\ &= \cdots = \frac{(2n)(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3} I(0). \end{aligned}$$

Now $I(0) = 1$, so

$$I(n) = \frac{(2n)(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3} = \frac{(2n)^2(2n-2)^2\cdots 2^2}{(2n+1)!} = \frac{2^{2n}(n!)^2}{(2n+1)!}$$

and thus

(3)
$$\sum_{j=0}^{n} \frac{(-1)^j}{2j+1}\binom{n}{j} = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

Exercise 2. For nonnegative integers n, let

$$J(n) = \int_0^1 (1+x^2)^n dx.$$

a. Show that

$$J(n) = \sum_{j=0}^{n} \frac{1}{2j+1}\binom{n}{j}.$$

b. Using the substitution $x = \tan\theta$, prove that

$$J(n) = \int_0^{\frac{\pi}{4}} \sec^{2n+2}\theta d\theta.$$

c. Use integration by parts to show

$$J(n) = \frac{2^n}{2n+1} + \frac{2n}{2n+1} J(n-1).$$

Hint: split up $\sec^{2n+2}\theta d\theta$ as $\sec^{2n}\theta d(\tan\theta)$.
d. Use the recurrence to find $J(3)$. (Note that $J(0) = 1$.)